Exercise 19

Prove the statement using the ε, δ definition of a limit.

$$\lim_{x\to 1} \frac{2+4x}{3} = 2$$

Solution

According to Definition 2, proving this limit is logically equivalent to proving that

$$\text{if} \quad |x-1| < \delta \qquad \text{then} \qquad \left|\frac{2+4x}{3} - 2\right| < \varepsilon$$

for all positive ε. Start by working backwards, looking for a number δ that's greater than $|x-1|$.

$$\left|\frac{2+4x}{3} - 2\right| < \varepsilon$$

$$\left|\frac{1}{3}[(2+4x) - 6]\right| < \varepsilon$$

$$\left|\frac{1}{3}(4x-4)\right| < \varepsilon$$

$$\left|\frac{4}{3}(x-1)\right| < \varepsilon$$

$$\frac{4}{3}|x-1| < \varepsilon$$

$$|x-1| < \frac{3\varepsilon}{4}$$

Choose $\delta = 3\varepsilon/4$. Now, assuming that $|x-1| < \delta$,

$$\begin{aligned}
\left|\frac{2+4x}{3} - 2\right| &= \left|\frac{1}{3}[(2+4x) - 6]\right| \\
&= \left|\frac{1}{3}(4x-4)\right| \\
&= \left|\frac{4}{3}(x-1)\right| \\
&= \frac{4}{3}|x-1| \\
&< \frac{4}{3}\delta = \frac{4}{3}\left(\frac{3\varepsilon}{4}\right) = \varepsilon.
\end{aligned}$$

Therefore, by the precise definition of a limit,

$$\lim_{x\to 1} \frac{2+4x}{3} = 2.$$